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On the Continuity of the Magnetization and Energy in Ising Ferromagnets

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We investigate the phase diagram of ferromagnetic Ising spin systems satisfying the G.H.S. inequality. We show that they cannot have a "normal" first-order phase transition as the temperature T is varied, i.e., one where the magnetization is discontinuous and there are three coexisting phases. Furthermore, for n.n. interactions, discontinuity in the magnetization at $0 < T_0 \leqslant T_c$ implies an uncountable infinity of pure phases at T_0.

KEY WORDS: Continuity magnetization; Ising ferromagnets; Thouless effect; infinity of pure phases.

1. INTRODUCTION

In this note we prove some new results concerning the phase diagram and the critical point of ferromagnetic Ising models with pair interactions:

$$\sigma_x = \pm 1, \qquad x \in \mathbf{Z}^d$$
$$H = -\sum_{x,y} J_{xy} \sigma_x \sigma_y - h \sum_x \sigma_x \tag{1}$$
$$0 \leqslant J_{xy} = J(|x-y|) \leqslant c/|x-y|^{d+\varepsilon}, \qquad \varepsilon > 0,\ J(1) > 0$$

While one might believe that everything interesting is already known about the behavior of such simple models, this is, unifortunately, still not the case. In particular at $h=0$ or $h \to 0$ and reciprocal temperature β larger

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than the critical $\beta_c = (k_B T_c)^{-1}$, there are still many unanswered questions; cf. Refs. 1–4 for review and references. β_c is defined here by the equivalent statements: (a) the spontaneous magnetization $m^*(\beta) = 0$ for $\beta < \beta_c$ and $m^*(\beta) > 0$ for $\beta > \beta_c$; (b) for $h = 0$ there is a unique Gibbs state for $\beta < \beta_c$ and at least two pure phases (translational invariant clustering states) for $\beta > \beta_c$. Clustering means $\langle \sigma_0 \sigma_x \rangle - \langle \sigma_0 \rangle \langle \sigma_x \rangle \to 0$ as $|x| \to \infty$ (which, by F.K.G. inequalities,[2] is equivalent to clustering of all correlation functions for any Gibbs state of our system).

We shall consider the following two questions for which we have some partial results:

(1) Is the spontaneous magnetization continuous in β for $\beta \geqslant \beta_c$? What about the continuity in β of other correlation functions like the (average) energy $\sum_x J_{0x} \langle \sigma_0 \sigma_x \rangle$?

(2) How many phases coexist for $\beta \geqslant \beta_c$?

For rapidly decaying interactions one certainly expects[1–4] continuity of $m^*(\beta)$ which implies continuity of all correlations and the existence of one state at β_c and two states for $\beta > \beta_c$. Nevertheless, the only exact results are the following: (a) for n.n. interactions continuity is known at all β for $d = 2$[1,2] and at β_c for $d \geqslant 4$.[5]

(b) for long-range interactions in one dimension

$$J_{xy} = \frac{c}{|x-y|^2} \qquad \text{for} \quad |x-y| \neq 0 \tag{2}$$

there is a discontinuity in the magnetization at β_c. This is the Thouless effect[6–8] which has now been proved rigorously.[9]

We also know that a discontinuity in $m^*(\beta)$ at β_0 implies that the susceptibility $\chi(\beta, \tilde{h})$, $\tilde{h} = \beta h$, must diverge as $\tilde{h} \to 0$, $\beta \uparrow \beta_0$.[2,3] In fact, if $\chi(\beta_0 - \delta, \tilde{h}) < \psi(\tilde{h})$, an integrable function of $\tilde{h}$ as $\tilde{h} \to 0$ or if $\lim_{\delta \downarrow 0} \delta \chi(\beta_0 - \delta, 0+) = 0$, then $m^*(\beta)$ must be continuous at β_0.[2] These results already rule out phase transitions in which $m^*(\beta)$ is discontinuous at β_0 yet fluctuations in all pure phases remain bounded as $\beta \to \beta_0$.

Our new results, which are stated and proven in Section 2, go further in this direction. In particular they show for the Thouless transition that if the state coming from the high-temperature side is a pure phase then there must be at β_c an uncountable infinity of pure phases. While the latter cannot be ruled out entirely it seems more likely[8] that there are just two states at β_c which is equivalent to the statement that the energy is continuous there.[2]

The results are discussed, with particular emphasis on the r^{-2} case, in Section 3 where we also present examples of similar models (spin-one Ising, Potts) which, however, not satisfying GHS, can and do have normal first-order transitions: one where the magnetization is discontinuous but there are no critical fluctuations.

2. RESULTS

We shall use three translation invariant Gibbs states $\langle\ \rangle^+$, $\langle\ \rangle^-$, and $\langle\ \rangle^0$ which are obtained as thermodynamic limits of finite-volume Gibbs states in a region $\Lambda \subset \mathbf{Z}^d$ with the following boundary conditions:

$$+\ \text{b.c., all } \sigma_x = +1, \qquad x \notin \Lambda$$

$$-\ \text{b.c., all } \sigma_x = -1, \qquad x \notin \Lambda$$

$$0\ \text{ b.c., no coupling between the spins in } \Lambda \text{ and in } \Lambda^C.$$

The existence of the thermodynamic limit for these states follow from Griffiths' inequalities[1] for the $\langle\sigma_A\rangle_\Lambda^{+,-,0}$ are monotone as $\Lambda \uparrow \mathbf{Z}^d$, $\sigma_A = \prod_{i\in A} \sigma_i$, $A \subset \Lambda$.

It also follows from these inequalities that the correlation functions $\langle\sigma_A\rangle^+$, $\langle\sigma_A\rangle^-$ are continuous in β as β *decreases* to β_0, while $\langle\sigma_A\rangle^0$ is continuous as β *increases* to β_0. Moreover, if $m^*(\beta) = \langle\sigma_x\rangle^+ = -\langle\sigma_x\rangle^-$ is continuous in β at β_0 then the energy, $\sum_x J_{0x}\langle\sigma_0\sigma_x\rangle^+$, and all correlation functions are continuous in β at β_0.[10] Continuity of the energy at β_0 (which, since the free energy is convex in β, holds for all values of β except possibly on a countable set) implies that $\langle\ \rangle^+$ and $\langle\ \rangle^-$ are the only pure phases at β_0,[10] in particular we have then $\langle\sigma_A\rangle^0 = \frac{1}{2}(\langle\sigma_A\rangle^+ + \langle\sigma_A\rangle^-)$ for all correlation functions. Finally, there is a unique phase at β_0 if and only if $m^*(\beta_0) = 0$.[11]

In order to state our main result, let us define the Gibbs state $\langle\ \rangle(\beta_0)$ at inverse temperature β_0 (without any superscripts) as

$$\langle\sigma_A\rangle(\beta_0) = \lim_{\beta\uparrow\beta_0} \langle\sigma_A\rangle^+(\beta)$$

The limits exist for all A's by monotonicity.[1,2]

Note that $\langle\sigma_A\rangle = \langle\sigma_A\rangle^0$ for $|A|$ even since $\langle\sigma_A\rangle^+ = \langle\sigma_A\rangle^0$ for almost all β if $|A|$ is even and $\langle\sigma_A\rangle^0$ is continuous as β increases to β_0. In particular, $\langle\ \rangle = \langle\ \rangle^0 = \langle\ \rangle^+$ ar any temperature where m^* vanishes, i.e., $\beta < \beta_c$.

Theorem 1. Let the magnetization be discontinuous at $\beta_0 (\geqslant \beta_c)$. Then if $\langle\ \rangle(\beta_0)$ is clustering there are an uncountable number of pure phases (translational invariant clustering Gibbs states) at β_0.

If we consider only nearest neighbour interactions in dimension $d \geqslant 3$ so that the infrared bounds[13] can be used, one can get a stronger version of Theorem 1:

Theorem 2. Let $d \geqslant 3$ and $J_{xy} = 0$ if $|x-y| \neq 1$. Then the states $\langle\ \rangle$ are clustering for all β. Furthermore, a discontinuity in $m^*(\beta)$ at some $\beta_0 (\geqslant \beta_c)$ implies an uncountable number of pure phases and, ipso facto, a discontinuity in the energy.

Proof of Theorem 1. Suppose that $m^*(\beta)$ is distontinuous at β_0,

$$\lim_{\beta \uparrow \beta_0} \langle \sigma_0 \rangle^+ (\beta) = \langle \sigma_0 \rangle \ \langle m^*(\beta_0) = \langle \sigma_0 \rangle^+ (\beta_0) \tag{3}$$

but that $\langle\ \rangle$ is clustering $[\langle\ \rangle = \langle\ \rangle(\beta)]$.

Let us now consider $h \neq 0$ in (1), and denote βh by h. For $\beta = 0$, $\langle \sigma_x \rangle(0, h) = m(0, h) = \tanh h$ and since $m(\beta, h)$ is jointly analytic in β and h (for $h \neq 0$),[14] we can find, for any $\gamma \in [0, 1]$ (by symmetry, the same is true for $-\gamma$) a curve $\beta_\gamma(h)$ in the (β, h) plane such that (see Fig. 1)

$$m(\beta_\gamma(h), h) = \gamma$$

$$\frac{d\beta_\gamma(h)}{dh} = -\frac{\partial m/\partial h}{\partial m/\partial \beta}$$

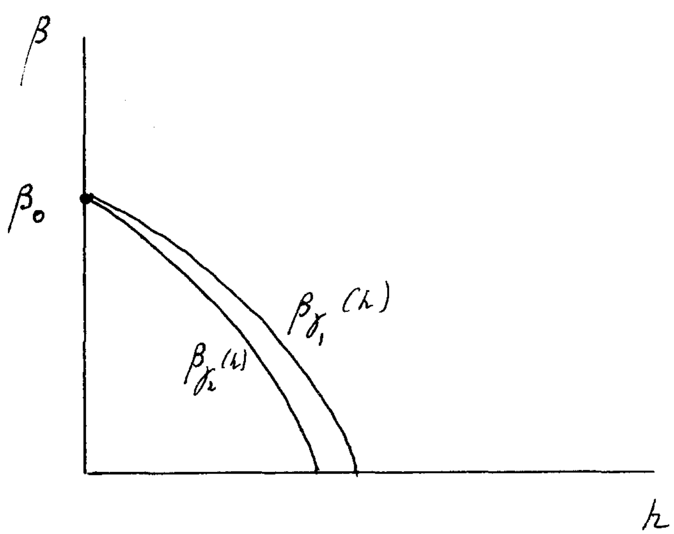

Fig. 1. Curves of constant magnetization; $\langle \sigma_0 \rangle < \gamma_2 < \gamma_1 < m^*(\beta_0)$.

no problem occurs unless $\partial m/\partial\beta = 0$ which, by inequalities, can only occur if $J_{xy} = 0$ all x, y.

When we let $h \to 0$ along the curve $\beta_\gamma(h)$, all correlations

$$\langle \sigma_A \rangle(\beta_\gamma(h), h) \to \langle \sigma_A \rangle^\gamma(\beta, 0)$$

[in particular $m(\beta_\gamma(h), h) \to \langle \sigma_0 \rangle^\gamma(\beta, 0)$] for some Gibbs state $\langle\ \rangle^\gamma$ at zero external field and inverse temperature $\beta = \lim_{h \to 0} \beta_\gamma(h)$. (This follows from the D.L.R. equations.)

Let us choose γ between $\langle \sigma_0 \rangle$ and $m^*(\beta_0)$. By the above construction, we obtain a (translation invariant) Gibbs state at $h = 0$ and $\beta = \beta_0$. We know that $\lim_{h \to 0} \beta_\gamma(h)$ must be β_0 because $m^*(\beta)$ is monotone increasing in β[9]

We also know that

$$\beta_\gamma(h) < \beta_0 \qquad \text{for all} \quad h > 0 \tag{4}$$

because one deduces from F.K.G. inequalities that $m(\beta, h)$ is increasing if we increase h and vary β along the line $\tilde{\beta}(h) = -\alpha^{-1} h + \beta_0$ (with $\alpha = \sum_x J_{0x}$),[15] so if $m(\beta_\gamma(h), h)$ is to be held constant, $\beta_\gamma(h) \leqslant \tilde{\beta}(h)$.

Now we use the G.H.S. inequality[12]: Write $\langle \sigma_0 ; \sigma_x \rangle \equiv \langle \sigma_0 \sigma_x \rangle - \langle \sigma_0 \rangle \langle \sigma_x \rangle$.

By G.H.S.

$$\langle \sigma_0 ; \sigma_x \rangle(\beta_\gamma(h), h) \leqslant \langle \sigma_0 ; \sigma_x \rangle(\beta_\gamma(h), 0) \tag{5}$$

(4) and (5) imply that

$$\begin{aligned} 0 &\leqslant \lim_{h \to 0} \langle \sigma_0 ; \sigma_x \rangle(\beta_\gamma(h), h) \\ &= \langle \sigma_0 ; \sigma_x \rangle^\gamma \\ &\leqslant \lim_{\beta \uparrow \beta_0} \langle \sigma_0 ; \sigma_x \rangle(\beta, 0) \\ &= \langle \sigma_0 ; \sigma_x \rangle \end{aligned}$$

which, by hypothesis, tends to zero as $|x| \to \infty$ ($\langle\ \rangle$ is clustering). Thus, for any γ between $\langle \sigma_0 \rangle$ and $m^*(\beta_0)$ we have that $\langle \sigma_0 ; \sigma_x \rangle^\gamma \to 0$ when $|x| \to \infty$. This implies, again by F.K.G. inequalities[2] (which hold for any Gibbs state of the Ising model), that all truncated correlation functions cluster and therefore $\langle\ \rangle^\gamma$ is a pure phase. Since, by construction $\langle \sigma_x \rangle^\gamma = \gamma$ we have obtained a continuum family of (different) ergodic Gibbs states at β_0.

Proof of Theorem 2. Using the infrared bounds[13] and correlation inequalities, Sokal[16] has shown that

$$\langle \sigma_{0;}\sigma_x \rangle^+(\beta) \leqslant \frac{C}{\beta |x|}$$

with $C < \infty$ independent of β. This bound also holds for limiting states like $\langle\ \rangle$.

3. DISCUSSION

(1) Theorems 1 and 2 hold for general models (higher spins, ϕ^4 lattice field theory) satisfying the G.H.S. inequality.[12,19]

(2) The content of the theorems is that systems satisfying the G.H.S. inequality cannot have a "normal" fist-order phase transition as β varies. Thus, in a situation where m^* is discontinuous and the energy continuous at $\beta_0 \geqslant \beta_c$ we must have continuity of the even correlations,[10] so

$$\lim_{\beta \uparrow \beta_0} \langle \sigma_0 \sigma_x \rangle^+(\beta) = \langle \sigma_0 \sigma_x \rangle^+(\beta_0) \geqslant [m^*(\beta_0)]^2 \tag{6}$$

where the last inequality follows fom Griffith.[1] Hence, for all $\Lambda \subset \mathbf{Z}^d$,

$$\lim_{\beta \uparrow \beta_0} |\Lambda|^{-1} \sum_{x, y \in \Lambda} [\langle \sigma_x \sigma_y \rangle(\beta) - [m^*(\beta)]^2 \geqslant c|\Lambda| \tag{7}$$

where $|\Lambda|$ is the number of sites in Λ and $c > 0$ is the discontinuity in $[m^*(\beta)]^2$ at β_0. This is a stronger divergence, with $|\Lambda|$, than that obtained at regular critical points where the magnetization is continuous. For "normal" first-order transitions the right-hand side of (6) would remain bounded as $\Lambda \to \infty$.

(3) For the r^{-2} potential in $d = 1$, the only case for which we know that there is a discontinuity in $m^*(\beta)$ at β_c we have no uniform apriory bound on $\langle \sigma_0 \sigma_x \rangle(\beta)$ for $\beta < \beta_c$. We are, therefore, left with several alternatives:

(a) Energy is continuous at β_c. This is what is expected[7,8] and Eqs. (6) and (7) then apply.

(b) If the energy is discontinuous then $\langle\ \rangle(\beta_c)$ cannot be a superposition of the + and − states[10] and we are left with these choices

(b_1) $\langle\ \rangle(\beta_c)$ is ergodic in which case Theorem 1 applies, or

(b_2) $\langle\ \rangle(\beta_c)$ is a superposition of some new kind of translation invariant[24] ergodic states which are neither + or −. All states $\langle\ \rangle^\gamma$ could then also be a superposition of such states which could be finite in number or there could still be an infinity of pure phases. In any case there cannot be exactly three pure phases at β_c since the third phase (in addition to the $+$ and $-$) would necessarily be $\langle\ \rangle$ (otherwise the energy would be continuous and there would be only two phases) and this would imply an infinity of phases.

We shall not pursue these speculations about the "sex of angels" any further here. Suffice it to say that unless there are an uncountable number of pure phases then as $\beta \uparrow \beta_c$, $\langle \sigma_0 \sigma_x \rangle(\beta)$ has *no* uniform bound $f(x)$ such that $f(x) \to 0$ as $x \to \infty$. This would explain the numerical observations.[8]

Some Examples

In this section we show, by means of examples, that one cannot prove too general statements about continuity of the energy and of the magnetization. Indeed we show that several models, having some inequalities in common with the Ising model, have quite a different behavior at their transition point:

(1) *Normal first-order transitions:* Consider the following spin-1 model: $S_x = 0, \pm 1$. The single-spin distribution is $a\delta(S_x) + (1-a)\,\delta(S_x^2 - 1)$ and the Hamiltonian is given by (1) with, say J_{xy} nearest neighbor. This is equivalent to the Blume–Capel[17] model (except that in this latter model a would depend on β). Then, for a close enough to 1, it has a first-order transition, with the magnetization and the energy discontinuous. At the transition temperature there are three pure phases $\langle\ \rangle^0$, $\langle\ \rangle^+$, and $\langle\ \rangle^-$ (see Ref. 18 for more details on this model). The G.H.S. inequality is not satisfied for a close to 1 ($a > \frac{2}{3}$; see Ref. 19). Actually, the states $\langle\ \rangle^\gamma$ used in the proof of Theorem 1 can still be constructed but turn out to be convex superpositions of the $\langle\ \rangle^+$ and $\langle\ \rangle^0$ states.

Another example of this type is the q-state Potts model which, for q large, is known[20] to have a first-order transition with $q+1$ phases at the transition point. We notice that for $q = 2^n$, n an integer, this Potts model is equivalent to an Ising model with ferromagnetic many-body interactions,[21] which again does not satisfy the G.H.S. inequalities.

(2) *Magnetization discontinuous and energy continuous:* As already mentioned, the $1/r^2$ Ising model in one dimensions is expected to be an example of this type. A trivial (degenerate) example of this phenomenon is the one dimensional Ising model with nearest-neighbor interactions at $\beta = \infty$ or $T = 0$:There $m^* = 0$ for all $T > 0$ but $m^* = 1$ for $T = 0$, so there is a jump in m^* at $T = 0$. However, the energy given by $\tanh \beta J$ is continuous, even C^∞, at the transition point ($T = 0$).

(3) *Magnetization continuous and energy discontinuous:* Although it is not proven, one excepts that the energy in Ising lattice gauge theories in dimension $d > 4$ is discontinuous at some temperature.[22] Since in this model the magnetization is always zero it is trivially continuous.

(4) *Uncountably many pure phases:* Our last example will show that the "pathological" alternative in Theorem 1 cannot be excluded on too general grounds. Consider a three-dimensional Ising model with no coupling between planes perpendicular to the z direction but ordinary nearest neighbour ferromagnetic couplings in each plane. At low temperatures we can obviously obtain any sequence of positively or negatively magnetized planes (since they are uncorrelated).

We construct uncountably many ergodic Gibbs states by distributing these + and − planes according to any ergodic measure on $\{-1, +1\}^Z$ (the set of Gibbs states of the one-dimensional Ising model, for fixed h and all possible temperatures, provides an uncountable family of such measures). Moreover, the energy and the magnetization here, being those of the two-dimensional Ising model, are perfectly continuous in β. This example is adapted from a similar example due to Slawny,[23] where all planes are coupled together, but through four-body interactions.

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